Workshop on differential calculus

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Introduction to the workshop

Welcome to the workshop in differential calculus! In this workshop we will start by introducing the derivative of a function and derive the most basic properties of it. We will also briefly introduce the notion of limits. The workshop has one primary purpose which is to obtain the understanding and the tools needed to calculate the derivative of a number of functions. Exercises account for a large part of the workshop. In the end we will consider a typical application of differential calculus, optimization.

The exercises have the difficulties indicated by blue dots, where one dot is easiest and three dots is the hardest

1 The derivative

1.1 Secant

The goal of this section is to introduce the *derivative*. It describes the change of a function in a point. To do this we first need to introduce some other concepts.

Definition 1.1. Let a function f be defined on an open interval (a,b), and let two points x_0 and x_1 be in (a,b). The *secant* of f between x_0 and x_1 is the straight line going through $f(x_0)$ and $f(x_1)$.

Remark 1.2. The gradient of the secant is given by

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0},$$

which is well-defined, as long as $x_1 \neq x_0$ (otherwise we get division by 0).

Example 1.3. Consider the function $f(x) = x^2 + x - 1$ and the two points $x_0 = -3$ and $x_1 = 4$. The graph below shows the function f (blue), the points $(x_0,f(x_0))$ and $(x_1,f(x_0))$ (red) as well as the secant through x_0 and x_1 (dark red).

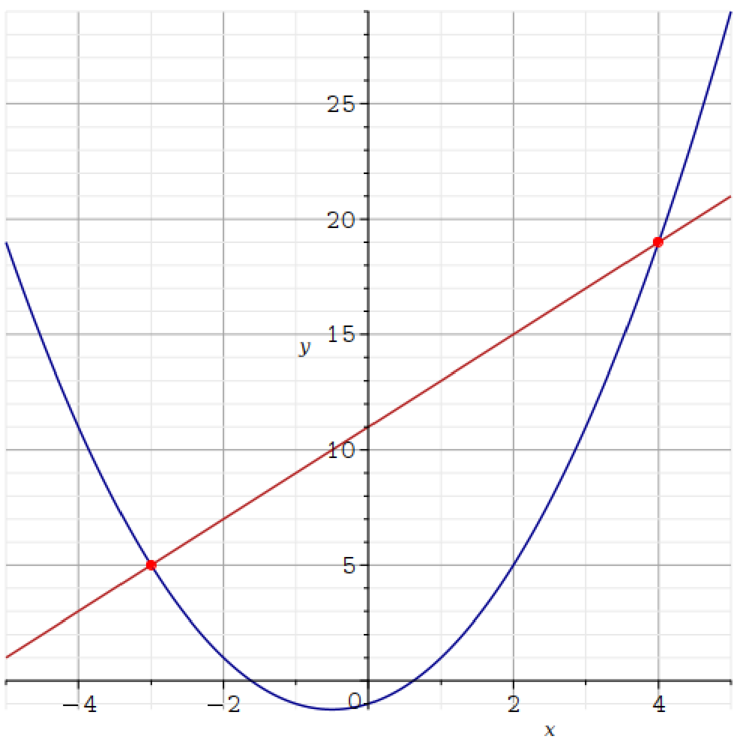


Figure 1: Illustration of a secant (dark red) for a second degree polynomial (blue).

We can determine the secant explicitly as follows. The gradient of the secant is given by

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(4) - f(-3)}{4 - (-3)} = \frac{19 - 5}{7} = 2,$$

and we can therefore determine the intersection with the y-axis b by using the fact that the secant touches the point $(4,f(4)) = (4,19)$. If we call the secant s we get the equation

$$s(4) = 19 \quad \Leftrightarrow \quad 19 = 2 \cdot 4 + b \quad \Leftrightarrow \quad b = 19 - 8 = 11.$$

Therefore the secant through -3 and 4 the expression $s(x) = 2x + 11$.

1.2 The difference quotient and the derivative

Let us once again consider the gradient of the secant between two points x_0 and x_1.

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

This value is also called the *difference quotient*. We can write the above in a different way. Let $\Delta x = x_1 - x_0$, were Δ is the Greek letter (capital) delta (Δ usually denotes some sort of difference). We note that $x_1 = x_0 + \Delta x$, and we can therefore write the difference quotient as

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

In differential calculus we are interested in describing the change of a function in a single point. We will do this by considering the difference quotient and see, what it goes toward when Δx goes to 0, corresponding to x_1 going towards x_0. More formally we wish to consider the limit

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x},$$

if it exists. Before we continue the discussion we will first study some examples that involve limits.

Example 1.4. Let us consider the function $g(x) = x + 3$. We wish to dertermine

$$\lim_{x \to 0} g(x).$$

We see that if x comes arbitrarily close to 0 then $x + 3$ will come closer to 3. Therefore we have

$$\lim_{x \to 0} g(x) = 3,$$

Which agrees with $g(0) = 3$

Example 1.5. Consider the fork function

$$g(x) = \begin{cases} -1, & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

The graph of the function is illustrated in the figure below:

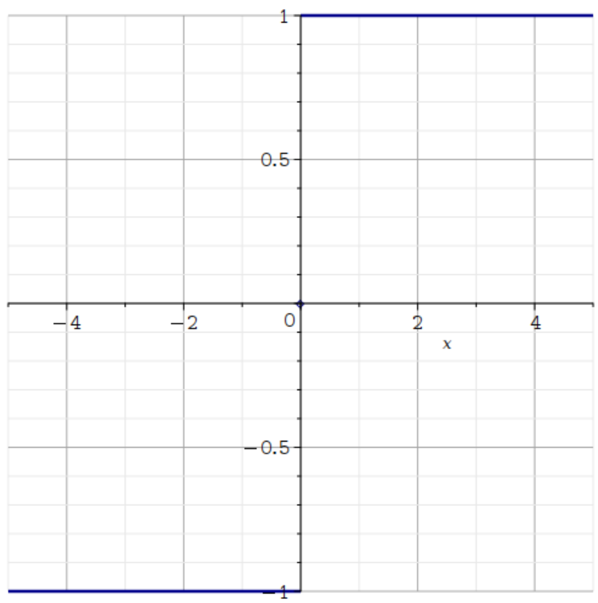


Figure 2: Example of a discontinous function.

In this case the limit

$$\lim_{x \to 0} g(x)$$

does not exist. The reason is that the function approaches something different, depending on what direction we approach 0. If we approach from the right the limit is 1. If we approach for the left the limit is -1

Example 1.6. Consider the function

$$h(x) = \begin{cases} 1 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}.$$

In this case we have

$$\lim_{x \to 0} h(x) = 1,$$

because the function approaches 1 no matter in what direction x approaches 0. Note that $h(0) = 0 \neq 1$. Therefore the limit is not always the same as evaluating the function in a point.

With a better understanding of limits we can return to the difference quotient and make the following definition.

Definition 1.7. Let f be a function defined on an open interval (a,b), and let x_0 be in (a,b). If the limit

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

exists, it is called the *derivative* in x_0 and is denoted $f'(x_0)$. $f'(x_0)$ is pronounced "f prime of x_0". If the derivative exists in x_0 we say that f is differentiable in x_0.

Example 1.8. Let $f(x) = a$ be a constant function and x_0 an arbitrary real number. The derivative is equal to

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{a - a}{\Delta x} = 0.$$

Here it is obvious that

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = 0.$$

So the derivative of a constant is equal to 0. This fits our intuition, as a constant function does not change in any points.

Example 1.9. Consider $f(x) = x^2$, and let x_0 be an arbitrary real number, we wish to determine $f'(x_0)$. We start by writing the difference quotient.

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{(x_0 + \Delta x)^2 - x_0^2}{\Delta x}.$$

We can now use the quadratic identities and get that the above is equal to

$$\frac{x_0^2 + (\Delta x)^2 + 2x_0\Delta x - x_0^2}{\Delta x} = \frac{(\Delta x)^2 + 2x_0\Delta x}{\Delta x} = \Delta x + 2x_0.$$

Lastly we consider the limit for $\Delta x \to 0$. It obviously goes to $2x_0$ in this limit. Therefore we have

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = 2x_0.$$

Below the function f is illustrated (blue) with the tangent through 1 drawn (red):

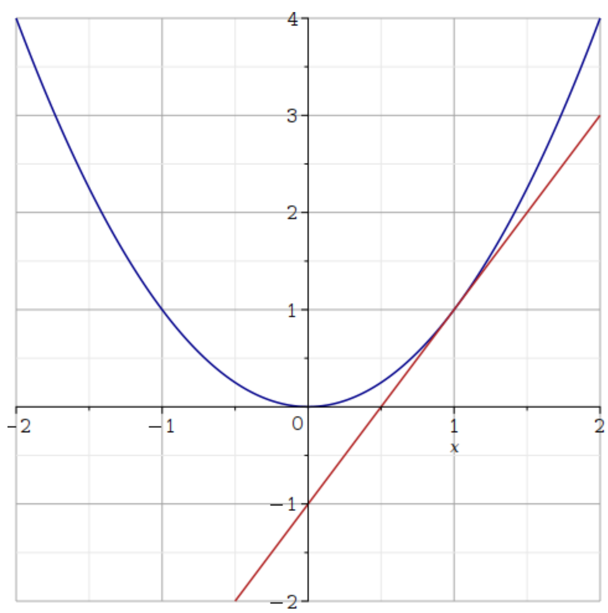


Figure 3: The function $f(x) = x^2$ (blue) with the tangent through $(1,f(1))$ drawn (red).

We can see that the line that is tangent to $(1,f(1))$ has a gradient of 2, which corresponds to $f'(1) = 2$. This is exactly the geometric interpretation of the derivative. It describes the gradient of the graph in a point.

1.3 Properties of the derivative

We have now seen some examples of how to calculate the derivative directly from the definition. This is however only a plausible strategy for very simple examples. For more complicated examples it is essential to know a few properties. We will consider them in this section.

Theorem 1.10. *Let f and g be functions that both are differentiable in x_0. We have that*

$$(f+g)'(x_0) = f'(x_0) + g'(x_0).$$

Proof. Remember that the function $f+g$ is defined by $(f+g)(x) = f(x) + g(x)$ We write the difference quotiont.

$$\begin{aligned}\frac{(f+g)(x_0+\Delta x) - (f+g)(x_0)}{\Delta x} &= \frac{f(x_0+\Delta x) + g(x_0+\Delta x) - (f(x_0) + g(x_0))}{\Delta x} \\ &= \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} + \frac{g(x_0+\Delta x) - g(x_0)}{\Delta x},\end{aligned}$$

and by taking the limit $\Delta x \to 0$ on both sides og the equal sign we get

$$(f+g)'(x_0) = f'(x_0) + g'(x_0)$$

as desired. ∎

Remark 1.11. We can prove that $(f-g)'(x_0) = f'(x_0) - g'(x_0)$ by a similar proof.

Example 1.12. Let $f(x) = x^2 + 5$. We know that the derivative of 5 is 0. We also know from before, that x^2 has the derivative $2x$. This gives us that $f'(x) = 2x + 0 = 2x$

We have seen that the derivative of the sum of functions is the sum of the derivatives. The situation is not quite as simple for the product of two functions.

Theorem 1.13 (The product rule). *Let f and g be functions that both are differentiable in a point x_0. Then we have*

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

Proof. Remember that the function fg is defined by $fg(x) = f(x)g(x)$. We write the difference quotient.

$$\frac{(fg)(x_0+\Delta x) - (fg)(x_0)}{\Delta x} = \frac{f(x_0+\Delta x)g(x_0+\Delta x) - f(x_0)g(x_0)}{\Delta x}.$$

It is not immediately obvious how to proceed from here. It requires a smart trick. We will add 0 in a smart way in the numerator. We get the idea to add $f(x_0)g(x_0+\Delta x) - f(x_0)g(x_0+\Delta x)$ to the numerator. Then the above becomes

$$\begin{aligned}&\frac{f(x_0+\Delta x)g(x_0+\Delta x) + f(x_0)g(x_0+\Delta x) - f(x_0)g(x_0+\Delta x) - f(x_0)g(x_0)}{\Delta x} = \\ &\frac{(f(x_0+\Delta x) - f(x_0))g(x_0+\Delta x) + f(x_0)(g(x_0+\Delta x) - g(x_0))}{\Delta x} \\ &\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}g(x_0+\Delta x) + f(x_0)\frac{g(x_0+\Delta x) - g(x_0)}{\Delta x}.\end{aligned}$$

We notice that we have rewritten the above expression to something involving the difference quotient of f and g in x_0 as well as $g(x_0 + \Delta x_0)$ and $f(x_0)$. Since g is differentiable in x_0 it is also continuos.

$$\lim_{\Delta x \to 0} g(x_0 + \Delta x) = g(x_0).$$

If we take the limit $\Delta x \to 0$ on both sides we get

$$f'(x_0)g(x_0) + f(x_0)g'(x_0),$$

which is what we wanted to show.

■

Theorem 1.14 (**The quotient rule**). *Let f and g be functions that both are differentiable in the point x_0. Assume that $g(x_0) \neq 0$. Then*

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

Proof. Remember that $(f/g)(x) = f(x)/g(x)$. We write the difference quotient

$$\frac{(f/g)(x_0+\Delta x) - (f/g)(x_0)}{\Delta x} = \frac{\frac{f(x_0+\Delta x)}{g(x_0+\Delta x)} - \frac{f(x_0)}{g(x_0)}}{\Delta x} = \frac{f(x_0+\Delta x)g(x_0) - f(x_0)g(x_0+\Delta x)}{\Delta x g(x_0)g(x_0+\Delta x)}.$$

The trick from here is similar to the trick from before. We add 0 to the numerator in a smart way, which is to add $f(x_0+\Delta x)g(x_0+\Delta x) - f(x_0+\Delta x)g(x_0+\Delta x)$ to the numerator. Then the difference quotient becomes

$$\frac{f(x_0+\Delta x)g(x_0) + f(x_0+\Delta x)g(x_0+\Delta x) - f(x_0+\Delta x)g(x_0+\Delta x) - f(x_0)g(x_0+\Delta x)}{\Delta x g(x_0)g(x_0+\Delta x)} =$$
$$\frac{(f(x_0+\Delta x) - f(x_0))g(x_0+\Delta x) - f(x_0+\Delta x)(g(x_0+\Delta x) - g(x_0))}{\Delta x g(x_0)g(x_0+\Delta x)} =$$
$$\left(\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}g(x_0+\Delta x) - f(x_0+\Delta x)\frac{g(x_0+\Delta x) - g(x_0)}{\Delta x}\right)\frac{1}{g(x_0)g(x_0+\Delta x)}.$$

If we use the continuity of f and g in x_0 we get

$$\lim_{\Delta x \to 0} f(x_0+\Delta x) = f(x_0) \quad \text{og} \quad \lim_{\Delta x \to 0} g(x_0+\Delta x) = g(x_0).$$

If we take the limit $\Delta x \to 0$ of the above expression we get

$$(f'(x_0)g(x_0) - f(x_0)g'(x_0))\frac{1}{g(x_0)^2} = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

This concludes the proof. ■

We will see plenty of examples of the above rules in the next section. We have expressed and proven the rules involving the basic operations. The only fundamental rule we miss is the *chain rule*.

Theorem 1.15 (**The chain rule**). *Let f be a function that is differentiable in the point x_0 and assume that g is a function that is differentiable in the point $f(x_0)$. The composition $g \circ f$ is differentiable in x_0 with derivative*

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

Remark 1.16. The theorem is understood as follows: To differentiate the composite function $(g \circ f)(x) = g(f(x))$ in a point x_0 we determine $g'(x)$ and substitute $f(x)$ for x and thereafter multiply with $f'(x_0)$.

Proof. We write the difference quotient.

$$\frac{(g\circ f)(x_0+\Delta x)-(g\circ f)(x_0)}{\Delta x}=\frac{g(f(x_0+\Delta x))-g(f(x_0))}{\Delta x},$$

and we get the idea to multiply with 1 in a smart way. We do it by multiplying and dividing with $f(x_0+\Delta x)-f(x_0)$ in the above. The difference quotient is then equal to

$$\frac{(g\circ f)(x_0+\Delta x)-(g\circ f)(x_0)}{\Delta x}=\frac{g(f(x_0+\Delta x))-g(f(x_0))}{f(x_0+\Delta x)-f(x_0)}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}.$$

Write $\Delta f=f(x_0+\Delta x)-f(x_0)$. The difference quotient is then

$$\frac{g(f(x_0+\Delta x))-g(f(x_0))}{\Delta f}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}.$$

The continuity of f in x_0 gives that

$$\lim_{\Delta x\to 0}\Delta f=\lim_{\Delta x\to 0}f(x_0+\Delta x)-\lim_{\Delta x\to 0}f(x_0)=f(x_0)-f(x_0)=0.$$

By using that g is differentiable in $f(x_0)$ we get

$$\lim_{\Delta x\to 0}\frac{g(f(x_0+\Delta x))-g(f(x_0))}{\Delta f}=\lim_{\Delta f\to 0}\frac{g(f(x_0+\Delta x))-g(f(x_0))}{\Delta f}=g'(f(x_0)).$$

So we get $g'(f(x_0))f'(x_0)$ by taking the limit $\Delta x\to 0$ of the difference quotient. Hence the proof is finished. ■

Example 1.17. Let $g(x)=x^2$ and $f(x)=x+2$. We wish to determine the derivative of $g(f(x))=(x+2)^2$. We have that $g'(x)=2x$ and $f'(x)=1$ (see the exercises). Thus we have

$$(g\circ f)'(x)=g'(f(x))f'(x)=2f(x)\cdot 1=2x+4.$$

This can also be done with direct calculation. We have $g(f(x))=x^2+4+4x$. If we differentiate by terms we will get the same answer.

In the next section we will introduce the derivative of a number of central functions. This will give us plenty of opportunity to use the results above.

1.4 Exercises

• **Exercise 1.1:**
Consider the function $f(x)=x^2-4$. Sketch the function and draw the secant through $x_0=-1$ and $x_1=2$.

•• **Exercise 1.2:**
Show directly from the definition of the derivative, that $f(x)=ax$ has the derivative $f'(x)=a$

•• **Exercise 1.3:**
Let f be a function that is differentiable in x_0. Show that the function $g(x)=af(x)$ is differentiable in x_0 with the derivative $g'(x_0)=af'(x)$.

••• **Exercise 1.4:**
Show directly from the definition that the derivative, that $f(x)=x^3$ has the derivative $f'(x)=3x^2$. Hint: here the cubic identities will be usefull: $(a+b)^3=a^3+3a^2b+3ab^2+b^3$.

2 Examples of derivatives

In this section we will look at some central functions and their derivatives

2.1 Polynomials

A polynomial is a function on the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where each a_i is a constant. If we wish to differentiate such a function, we can differentiate each term and then add all the derivatives using 1.10. From exercise 1.3 we have that a constant multiplied with a function does not change its derivative. This means that we only need to study the function on the form x^n, where n is a positive integer. We have the following result

Theorem 2.1. *The function $f(x) = x^n$ for an integer $n \geq 1$ has the derivative $f'(x) = nx^{n-1}$.*

Proof. The proof builds on the observation that $(x_0+\Delta x)^n$ is equal to $x_0^n+nx_0^{n-1}\Delta x+$(terms that have $(\Delta x)^2$ multiplied with something else). When we write the difference quotient we divide by Δx and it therefore becomes $nx^{n-1}+$(terms that have Δx multiplied with something else). If we let Δx approach 0 we only have nx^{n-1} left. ∎

Example 2.2. Let $f(x) = 5x^4 - 3x^2 + 8x + 10$. The first term has the derivative $5 \cdot 4x^{4-1} = 20x^3$. The second terms has the derivative $3 \cdot 2x^{2-1} = 6x$, while the third term has the derivative 8. The last term is constant and thus has the derivative 0. All together it gives us

$$f'(x) = 20x^3 - 6x + 8.$$

2.2 Trigonometric functions

In this section we will consider the trigonometric functions cos and sin. We start by recalling the *addition formulas*, which we do not show.

Theorem 2.3 (The addition formulas). *The following relations hold:*

$$\begin{aligned}
\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\
\sin(x+y) &= \cos(x)\sin(y) + \sin(x)\cos(y).
\end{aligned}$$

Theorem 2.4. *The functions* cos *and* sin *are both differentiable in all points with*

$$\cos'(x) = -\sin(x) \quad og \quad \sin'(x) = \cos(x).$$

Proof. We write the difference quotient for cos and use the addition formulas:

$$\begin{aligned}
\frac{\cos(x_0+\Delta x) - \cos(x_0)}{\Delta x} &= \frac{\cos(x_0)\cos(\Delta x) - \sin(x_0)\sin(\Delta x_0) - \cos(x_0)}{\Delta x} \\
&= \cos(x_0)\frac{\cos(\Delta x) - 1}{\Delta x} - \sin(x_0)\frac{\sin(\Delta x)}{\Delta x}.
\end{aligned}$$

We will now use the two results

$$\lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x} = 0 \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} = 1.$$

If we take the limit $\Delta x \to 0$ of the difference quotient we get $-\sin(x_0)$. Hence we have

$$\cos'(x_0) = -\sin(x_0),$$

Which shows the first statement. We will now consider the derivative of sin. We write the difference quotient and use the addition formulas:

$$\begin{aligned} \frac{\sin(x_0 + \Delta x) - \sin(x_0)}{\Delta x} &= \frac{\cos(x_0)\sin(\Delta x) + \sin(x_0)\cos(\Delta x) - \sin(x_0)}{\Delta x} \\ &= \cos(x_0)\frac{\sin(\Delta x)}{\Delta x} + \sin(x_0)\frac{\cos(\Delta x) - 1}{\Delta x}. \end{aligned}$$

If we use the same results as above, we get $\sin'(x_0) = \cos(x_0)$.

■

Remark 2.5. The two limits in the proof above are not trivial. To prove them takes a formal introduction to cos and sin as well as establishing some central inequalities. Furthermore it requires a more formal definition of the notion of a limit, which we will not cover.

We are now ready to do more complex differentiation problems.

Example 2.6. Consider the function $f(x) = x^3\cos(x)$. We wish to determine $f'(x)$. We start by noting that $f(x)$ is the product of $g(x) = x^3$ and $\cos(x)$. We know that $g'(x) = 3x^2$ and $\cos'(x) = -\sin(x)$. With the product rule we get

$$f'(x) = g'(x)\cos(x) + g(x)\cos'(x) = 3x^2\cos(x) - x^3\sin(x).$$

Example 2.7. Consider the function $f(x) = \sin(4x^2)$. We wish to determine $f'(x)$. We notice that $f(x) = h(g(x))$ where $g(x) = 4x^2$ and $h(x) = \sin(x)$. We know that $h'(x) = \cos(x)$ and $g'(x) = 8x$. With the chain rule we get that

$$f'(x) = h'(g(x))g'(x) = 8x\cos(4x^2).$$

2.3 The exponential function and the natural logarithm

We will now consider the exponential function e^x and the natural logarithm $\ln(x)$. The following result (where we omit the proof) gives the derivative of these two functions.

Theorem 2.8. *We have*

$$(e^x)' = e^x \quad \textit{and} \quad \ln'(x) = \frac{1}{x}.$$

Notice that $(e^x)' = e^x$ tells us that the change of the exponential function in an arbitrary point is equal to the value of the function in that point. This is a particularly interesting property.

Example 2.9. Let $f(x) = \ln(x^2 + 1)$. By using the chain rule we get

$$f'(x) = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}.$$

2.4 Review of rules

Function	$f'(x)$	Note
$f+g$	$(f+g)'(x) = f'(x) + g'(x)$	
fg	$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$	
f/g	$(f/g)'(x) = (f'(x)g(x) - f(x)g'(x))/g(x)^2$	$g(x) \neq 0$

Figure 4: Table over rules for differentiation

$f(x)$	$f'(x)$	Note
a	0	a constant
x^n	nx^{n-1}	$n \geq 1$ an integer
$\cos(x)$	$-\sin(x)$	
$\sin(x)$	$\cos(x)$	
$\tan(x)$	$1 + \tan(x)^2$	
e^x	e^x	
$\ln(x)$	$1/x$	

Figure 5: Table over some important functions and their derivatives

2.5 Exercises

• **Exercise 2.1:**

Determine the derivative of the following polynomials:

1) $6x^2 + 7$.

2) $3x^4 - 7x^3 + 10x - 11$.

3) $3x^7 + 12x^5$.

• **Exercise 2.2:**

Determine the derivative of the following functions:

1) $4\cos(x) + \sin(x)$.

2) $e^x + 2\sin(x)$.

3) $x^2 + x + \ln(x)$.

•• **Exercise 2.3: Product rule**

Determine the derivative of the following functions:

1) $2x^2\cos(x)$.

2) $\cos(x)\sin(x)$.

3) $4x^3e^x$.

4) $\cos(x)e^x$.

5) $x^2\cos(x)\sin(x)$.

6) $\sin(x)\ln(x)$.

•• **Exercise 2.4: Quotient rule**

Determine the derivative of the following functions:

1) $\frac{x+1}{x^3}$.

2) $\frac{\cos(x)}{\sin(x)}$.

3) $\frac{\cos(x)}{x^2}$.

4) $\frac{\ln(x)}{x^2}$.

5) $\frac{e^x}{4x+1}$.

6) $\frac{x^3}{2e^x+4}$.

••• **Exercise 2.5: Chain rule**

Determine the derivative of the following functions:

1) $\cos(x^3)$.

2) $\sin(\cos(x))$.

3) $\cos(e^x)$.

4) $e^{\cos(x)+1}$.

5) $\ln(e^x)$.

6) e^{x^2}.

7) $\sin(e^{2x} + 10x)$.

8) $\cos(x)^7$.

9) $\ln(x)^3$.

10) $\ln(\cos(x))$.

11) $\tan(x^2)$.

●● **Exercise 2.6:**
Recall that $\tan(x) = \sin(x)/\cos(x)$. Below $''$ and $'''$ is understood as differentiating two and three times respectively.

1)Show that $\tan'(x) = 1 + \tan(x)^2$.

2)Calculate $\tan''(x)$ and $\tan'''(x)$.

●● **Exercise 2.7: Hyperbolic functions**
In this exercise we will look at the hyperbolic functions. *Hyperbolic cosine* and *hyperbolic sine* are given by

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

1)show that $\cosh'(x) = \sinh(x)$, and that $\sinh'(x) = \cosh(x)$.

2)We define *hyperbolic tangent* as

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}.$$

Determine $\tanh'(x)$ og $\tanh''(x)$.

●●● **Exercise 2.8: Mixed exercises**
Determine the derivative of the following functions:

1) $\frac{\cos(e^x)+x^2}{\sin(x)}$.

2) $e^{\cos(x)\sin(x)}$.

3) e^{e^x}.

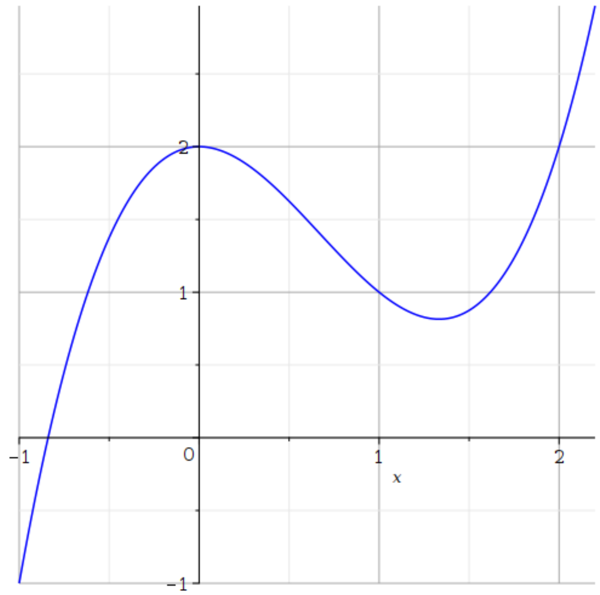
4) $x^3e^x - \frac{\cos(x)}{x}$.

5) $\cos(x)\tan(x)$.

3 An application: optimization

3.1 Maxima and minima

Differential calculus has a plethora of applications. We will consider a relatively simple application, optimisation. An example of an optimization problem could be that you wish to make a container in a specific shape, and you wish to maximise the volume compared to a given area. Generally in an optimization problem there is a function that we need to find the minimum or maximum of. To see how differential calculus plays a role in finding minima or maxima we consider the graph for the function $f(x) = x^3 - 2x^2 + 2$ as an example:



On the figure it appears that the function has two extrema (a general term for maxima and minima), which are a maximum and a minimum. What x-values do these points have? Lets recall what the derivative tells us. The derivative shows the gradient in a point. In a maximum or a minimum the gradient will be zero, because if it was greater than zero, there would be a point immediately close by where the value of the function would be greater or smaller. We conclude that the x value of an extrema satisfies.

$$f'(x) = 0.$$

Lets find extrema for the function f. We differentiate and get

$$f'(x) = 3x^2 - 4x.$$

We thus have to solve the equation $3x^2 - 4x = 0$ for x. We see that $x = 0$ is a solution. To find the other solution we assume that $x \neq 0$. We divide by x on both sides of the equal sign and get $3x - 4 = 0$ which has the solution $x = 4/3$. So f has extrema in $x = 0$ and $x = 4/3$. We can see on the graph that $x = 0$ is a (local) maximum, while $x = 4/3$ is a (local minimum). But what if we couldn't see the graph?

It is fortunately easy to check whether an extremum is a minimum or a maximum by looking at the second derivative $f''(x)$. The second derivative describes the gradient of the gradient. Or with other words, how the gradient og f changes. Suppose that x is a minimum, then the gradient will go from negative to positive, so $f''(x)$ is positive. On the other hand if x is a maximum, the graph goes down after x, so $f''(x)$ is negative. For our concrete function we have $f''(x) = 6x - 4$. So $f''(0) = -4 < 0$, and $f''(4/3) = 2 > 0$. This aligns with what we saw on the graph.

Lets recap the method to find extrema: Given a function f that is two times differentiable (f and f' are differentiable) we can find the extrema by solving the equation $f'(x) = 0$. If $f''(x) < 0$ the point is a maximum, while $f''(x) > 0$ shows that the point is a minimum.

Example 3.1. Suppose that we have 20 m^2 steel available and wish to make a box-shaped container with as great a volume as possible. The height and the width should be the same. Let x be the height/ width and y the length. The area is $A(x,y) = 2x^2 + 4xy$, while the volume is $V(x,y) = x^2y$. We know that $A(x,y) = 20$. We then have the equation $20 = 2x^2 + 4xy$, which we will write as $10 = x^2 + 2xy$. We isolate y

$$10 = x^2 + 2xy \quad \Leftrightarrow \quad 10 - x^2 = 2xy \quad \Leftrightarrow \quad y = \frac{10 - x^2}{2x}.$$

We can insert this expression for y in the function for the volume, and thereby the volume can be expressed as a function of x alone

$$V(x) = x^2\frac{10 - x^2}{2x} = \frac{1}{2}x(10 - x^2) = 5x - \frac{1}{2}x^3.$$

We differentiate this function and get

$$V'(x) = 5 - \frac{3}{2}x^2.$$

We solve $V'(x) = 0$ for x:

$$0 = 5 - \frac{3}{2}x^2 \quad \Leftrightarrow \quad 5 = \frac{3}{2}x^2 \quad \Leftrightarrow \quad \frac{10}{3} = x^2.$$

x can not be negative, so the solution is $x = \sqrt{10/3} \approx 1.826$. Lets investigate whether this is a maximum for volume. We have

$$V''(x) = -3x,$$

which is negative for all positive values of x. Thus $x = \sqrt{10/3}$ is a maximum. Then the maximal volume becomes $V(\sqrt{10/3}) \approx 6.086m^3$

3.2 Exercises

●● **Exercise 3.1:**
Determine all local maxima and minima for the function $f(x) = x^3 + 2x^2 - 4x + 2$.

●●● **Exercise 3.2:**
We wish to make an enclosure for a garden. We have 40m of fence available, and want the garden to be rectangular. Determine the optimal length and width of the garden as well as the greatest possible area.